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ON THE LINEAR PREDICTION THEORY OF STATIONARY PROCESSES INDEXED BY LATTICE POINTS*

by

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ERRATA SHEET for Technical Report 100

Page	Line	Reads	Should Read
2	2	group	subgroup
3	2	$\bigcap_{t=-\infty}^{\infty} \mathcal{M}_t = \{0\}$	$\bigcap_{t=-\infty}^{0} \mathcal{M}_t = \{0\}$
3	13	$\mathcal{N}$	$\mathcal{N}_t$
3	11*	$\bigcap_{-\infty<t<\infty} \mathcal{N}_t = \{0\}$	$\bigcap_{-\infty<t\leq 0} \mathcal{N}_t = \{0\}$
4	2	$t \in R$	$t \in R,$
4	4	(ii) $\bigcap \mathcal{N}_t = \{0\}$	(ii) $\bigcap_{t<0} \mathcal{N}_t = \{0\}$
4	5	$\bigcup \mathcal{N}_t$	$\bigcup_{-\infty<t<\infty} \mathcal{N}_t$
5	8	$I_A(x) = 0$ or 1	$I_A(x) = 1$ or 0
5	14	reals	group of real numbers
6	2	$= \chi_p(\varphi(s))V_s\int \chi_p(x)\beta(dx)$	$= \chi_p(\varphi(s))V_s \int_{\Gamma} \chi_p(x)\beta(dx)$
6	5	U_pV_S	U_pV_s
7	2	$\mu_t(A) = \mu(A + \varphi(T))$	$\mu_t(A) = \mu(A + \varphi(t))$
8	5*	so	. So
11	4	as to be	to be
14	3*	$= \frac{B(x+\varphi(x_0)+\varphi(s))}{B(x+\varphi(x_0))}$	$= \frac{B(x+\varphi(r_0)+\varphi(t)+\varphi(s))}{B(x+\varphi(r_0)+\varphi(t))}$
17	8	$\mu(A\Delta A+p)$	$\mu(A \Delta [A+p])$
19	1	$E(\sigma \cap A + p)$	$E([\sigma \cap A] + p)$
19	4*	$\bigcap \mathcal{N}_t = \{0\}$	$\bigcap_{-\infty<t<\infty} \mathcal{N}_t = \{0\}$
21	5, 6, 8	signed	complex
21	5.1(c)	$\bigcap \mathcal{N}_s = \{0\}$	$\bigcap_{-\infty<s<\infty} \mathcal{N}_s = \{0\}$
22	1	main	Main
22	2	[1]	in [1]

Page	Line	Reads	Should Read
23	10*	Let denote	Let $\mathcal{F}$ denote
23	9*	$L_2(S \times R, \quad, \nu \times \ell)$	$L_2(S \times R, \mathcal{F}, \nu \times \ell)$
23	2*	defined by	is defined by
26	10	$J\mathcal{H}_2$	$J_1\mathcal{H}_2$
26	8*	$\bigcap_p \mathcal{K}_p = \{0\}$	$\bigcap_{-\infty<p<\infty} \mathcal{K}_p = \{0\}$
29	7,8	$\bigcap_t \mathcal{N}_t$	$\bigcap_{-\infty<t<\infty} \mathcal{N}_t$

Introduction. The prediction theory of weakly stationary stochastic processes indexed by a dense subgroup of the group of real numbers was initiated by H. Helson and D. Lowdenslager in their papers ([5], [6]). In [6], analytic necessary and sufficient conditions for pure non-determinism of the processes are obtained when the spectrum of the process is absolutely continuous with respect to the Haar measure on the compact dual of the indexing subgroup (regarded as topologically discrete group). In this paper we shall study the properties of the spectrum without assuming that it is absolutely continuous with respect to the Haar measure. The paper is divided into two parts. The first consists of sections 2, 3 and 4. In sections 2 and 4 we study, via Weyl-Von Neumann equations, the qualitative properties and interrelation between the dual spectral measures associated with the process. In section 3 we exhibit a class of measures which apparently includes the most general form of measures singular with respect to the Haar measure and corresponding to a purely non-deterministic process (type 3(c) section 3).

In the second part, consisting of sections 5 to 8, we study the prediction problem of purely non-deterministic processes corresponding to the singular measures of the type described. A solution for this case is obtained similar to the one given by Helson and Lowdenslager in ([6], [7]) for the absolutely continuous case.

In section 1 we give the basic definitions and notations. For the sake of definiteness and simplicity we have taken the indexing subgroup to be $\{m+\lambda n : \lambda \text{ irrational, } m, n, \text{ integers}\}$ but this is not a restriction ([4]) and our results apply more generally for arbitrary countable dense subgroup on the real line.

<u>Section 1</u>. Let λ be an irrational number. Let L be the additive group of real numbers given by $L = \{m+\lambda n: \; m,n \text{ integers}\}$. Since λ is irrational L is dense in the additive group of real numbers. We regard L as a topologically discrete group. The map $\psi: (m,n) \to m+\lambda n$ is homeomorphic isomorphism of the group of lattice points in the plane onto the group L with the discrete topology. The dual T of L is homeomorphically isomorphic to the torus group. We shall denote by p, q, etc. the typical elements $m+\lambda n$, $m'+\lambda n'$ of L. χ_p, χ_q, etc. shall denote the characters on T corresponding to $p \in L$. We shall always denote by $\mathcal{A}$, the Borel subsets of T and by $\mathcal{B}$ the Borel subsets of the real line under usual topology.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X_p, $p \in L$, be a stochastic processes indexed by L, X_p being a random variable defined on Ω. We say that X_p, $p \in L$, is weakly stationary if $\mathcal{E}|X_p|^2 < \infty$ for each $p \in L$ and if $\mathcal{E} X_p \overline{X}_q = \mathcal{E} X_{p-q} \overline{X}_0$. It is easy to see that $r(p) = \mathcal{E} X_p \overline{X}_0$ is a positive definite function on L. So by Bochner's theorem for groups there exists a finite non-negative regular measure μ on T such that $r(p) = \int_T \chi_p(x)\mu(dx)$. Let us write $SX_p = \chi_p$, $p \in L$, then it is easy to show that S extends by linearity to an invertible isometry from H, the subspace of $L_2(\Omega)$, spanned by $\{X_p: p \in L\}$ onto $L_2(T,\mu)$. For each real number t let $\mathcal{M}_t$ = subspace of H spanned by $\{X_p : p \le t\}$ and $\mathcal{N}_t = S\mathcal{M}_t$. Let τ^p, $p \in L$, be the group of unitary operators on H defined by $\tau^p X_q = X_{p+q}$ and write $U^p = S\tau^p S^{-1}$. Following relations are easy to verify (i) $U^p f = \chi_p f$, $f \in L_2(T,\mu)$

(ii) $\mathcal{N}_t$ = subspace of $L_2(T,\mu)$ spanned by $\{\chi_p : p \le t\}$

(iii) $\tau^p \mathcal{M}_q = \mathcal{M}_{p+q}$ $\quad U^p \mathcal{N}_q = \mathcal{N}_{p+q}$.

In terms of the subspaces $\mathcal{M}_t$ we give the following definitions

Definition 1.1. A weakly stationary stochastic processes X_p, $p \in L$, is called purely non-deterministic if and only if $\bigcap_{t=-\infty}^{\infty} \mathcal{M}_t = \{0\}$.

Definition 1.2. A purely non-deterministic process X_p, $p \in L$ is called evanescent if $\bigcup_{t<0} \mathcal{M}_t$ is dense in $\mathcal{M}_0$, i.e., if X_0 is in the subspace spanned by $\{X_p;\ p < 0\}$.

Since $S\mathcal{M}_t = \mathcal{N}_t$, the definition of purely "non-deterministic" and "evanescent processes" have the following obvious reformulation in terms of the subspaces $\mathcal{N}_t$. The process $\{X_p,\ p \in L\}$ is purely non-deterministic if and only if $\bigcap_{-\infty<t<\infty} \mathcal{N}_t = \{0\}$ and it is evanescent if and only if $\bigcup_{t<0} \mathcal{N}_t$ is dense in $\mathcal{N}_0$, i.e., if the function 1 lies in the subspace of $L_2(T,\mu)$ spanned by $\{\chi_p; p < 0\}$.

Thus, in our context, the study of purely non-deterministic process is equivalent to the study of a family of subspaces $\mathcal{N}$ of $L_2(T,\mu)$ such that

(i) $\bigcap_{-\infty<t<\infty} \mathcal{N}_t = \{0\}$

(ii) $\mathcal{N}_p = U_p \mathcal{N}_0$ for $p \in L$

(iii) $\bigcup_{-\infty<t<\infty} \mathcal{N}_t$ is dense in $L_2(T,\mu)$

(iv) $\mathcal{N}_t$ = subspace of $L_2(T,\mu)$ spanned by $\{\chi_p:\ p \leq t\}$.

For this reason we shall henceforth deal exclusively with $L_2(T,\mu)$ and the appropriate subspaces $\mathcal{N}_t$ of it. In the next section we shall give properties of measure μ in order that a family of subspaces $\mathcal{N}_t$ satisfying (i) (ii) (iii) exists in $L_2(T,\mu)$. We shall also give a characterization of the subspaces $\mathcal{N}_t$. Our method here derives principally from the work of H. Helson and D. Lowdenslager [7], F. Forelli [3], K. deLeeuw and I. Glicksberg [1] and our paper [8].

Section 2. Let $L_2(T,\mu)$ and U_p, $p \in L$, be as in section 1. Let $\mathcal{N}_t$, $t \in R$ be a family of subspaces in $L_2(T,\mu)$ such that

(i) $\mathcal{N}_p = U_p \mathcal{N}_0 \quad p \in L$

(ii) $\bigcap \mathcal{N}_t = \{0\}$

(iii) $\bigcup \mathcal{N}_t$ is dense in $L_2(T,\mu)$.

In this section we shall study and give a characterization of such subspaces. First we note, however, that condition (iii) is not a serious restriction. For the study of a family of subspaces satisfying (i) and (ii) can be reduced to the study of the family of subspaces satisfying (i) (ii) and (iii) by following two observations:

(A) $\bigcup_{-\infty<t<\infty} \mathcal{N}_t$ spans a subspace of $L_2(T,\mu)$ which is invariant under U_p, i.e., under multiplication by χ_p, $p \in L$.

(B) If a subspace $\mathcal{K}$ of $L_2(T,\mu)$ is invariant under multiplication by every χ_p, $p \in L$, then there exists a Borel set $A \in \mathfrak{A}$ such that $\mathcal{K} = L_2(T,\mu|_A)$, where $\mu|_A$ means the measure μ restricted to A.

Each subspace $\mathcal{N}_t$ is simply invariant in the sense that $U_p \mathcal{N}_t = \mathcal{N}_{p+t} \subset \mathcal{N}_t$ whenever $p \leq 0$ but for any $p > 0$ $U_p \mathcal{N}_t \not\subset \mathcal{N}_t$. The subspaces $\mathcal{N}_t$ are therefore increasing, i.e., $\mathcal{N}_t \subsetneqq \mathcal{N}_s$ for $t < s$. They give rise to a spectral measure E on the Borel subsets $\mathcal{B}$ of R. If $(a,b]$ is the interval $a < x \leq b$, then $E(a,b] =$ orthogonal projection on $\mathcal{N}_b \ominus \mathcal{N}_a$. Since $\mathcal{N}_p = U_p \mathcal{N}_0$, E satisfies for all $B \in \mathcal{B}$ and $p \in L$, the relation $U_p E(B) = E(B+p) U_p$, which is $U_p E(B) U_{-p} = E(B+p)$. Conversely, given a spectral measure E on $\mathcal{B}$ with values orthogonal projections in $L_2(T,\mu)$ and satisfying $U_p E(B) U_{-p} = E(B+p)$, $p \in L$, $B \in \mathcal{B}$, we write $\mathcal{N}_t = E(-\infty,t] L_2(T,\mu)$.

Then the family of subspaces $\mathcal{N}_t$ satisfies (i) (ii) and (iii). Thus there is a one-one correspondence between the family of subspaces $\mathcal{N}_t$ satisfying (i) (ii) and (iii) and the spectral measure E satisfying $U_p E(B) U_{-p} = E(B+p)$, $B \in \mathcal{B}$, $p \in L$.

The group of unitary operators U_p, $p \in L$ has the spectral representation $U_p = \int_T \chi_p(x)\beta(dx)$ where β is the spectral measure on the Borel subsets of T defined by $\beta(A)f = I_A f$, $f \in L_2(T,\mu)$ $A \in \mathcal{A}$ and $I_A(x) = 0$ or 1 according as $x \in A$ or $x \notin A$ (see [9], 392). Let V_s, $s \in R$ be the group of unitary operators defined by $V_s = \int_{-\infty}^{+\infty} e^{is\lambda} E(d\lambda)$. We need the following imbedding of R in T (see [1]) for further study of the relation between V_s and U_p.

For each real r, let $\varphi(r)$ denote the character on L defined by $\varphi(r)(p) = e^{irp}$. If $p = m+\lambda n$, for example, then $\varphi(r)(p) = e^{ir(m+\lambda n)}$. It is easy to see that φ is a continuous isomorphism of the reals with the usual topology into T. Further, $\varphi(R)$ is dense in T. Geometrically $\varphi(R)$ is the line $y = \lambda x$ folded inside the square $0 \le x, y \le 2\pi$ by identifying every point (x,y) in the plane with $(x+2m\pi, y+2n\pi)$, m,n being integers. The isomorphism φ and the group $\varphi(R)$ will play an important role in our discussions. The main result of this section is the following (see [3], [8])

Theorem 2.1. The following three conditions are equivalent:

(a) $U_p V_s = \chi_p(\varphi(s)) V_s U_p$;

(b) $V_s \beta(A) V_{-s} = \beta(A+\varphi(s))$ $A \in \mathcal{A}$, $s \in R$;

(c) $U_p E(B) U_{-p} = E(B+p)$, $B \in \mathcal{B}$, $p \in L$.

Proof

$$U_p V_s = \int_T \chi_p(x)\beta(dx) V_s \ . \tag{2.1}$$

But

$$(2.2)\qquad \chi_p(\varphi(s))V_sU_p = \chi_p(\varphi(s))V_s \int \chi_p(x)\beta(dx)$$

$$= \int_T \chi_p(x+\varphi(s))V_s\beta(dx)$$

$$= \int_T \chi_p(x)V_s\beta(dx-\varphi(s)) \; .$$

Since $U_pV_S = \chi_p(\varphi(s))V_sU_p$, we can equate (2.1) and (2.2) to obtain

$$\int_T \chi_p(x)\beta(dx)V_s = \int_T \chi_p(x)V_s\beta(dx-\varphi(s)) \; ,$$

i.e., for all $f, g \in L_2(T,\mu)$,

$$\int_T \chi_p(x)(\beta(dx)V_sf,g) = \int_T \chi_p(x)(V_s\beta(dx-\varphi(s))f,g) \; .$$

By the uniqueness of Fourier transform it follows that for all $f,g \in L_2(T,\mu)$ $(\beta(A)V_sf,g) = (V_s\beta(A-\varphi(s))f,g)$; i.e., $\beta(A)V_s = V_s\beta(A-\varphi(s))$ or equivalently, $V_s\beta(A)V_{-s} = \beta(A+\varphi(s))$. This proves that (a) implies (b). The same method shows that (a) implies (c) provided we substitute the spectral representation of V_s instead of that of U_p.

(b) implies (a)

$$U_pV_s = \int_T \chi_p(x)\beta(dx)V_s = \int_T \chi_p(x)V_s\beta(dx-\varphi(s))$$

$$= V_s \int_T \chi_p(x)\beta(dx-\varphi(s))$$

$$= V_s\chi_p(\varphi(s))\int_T \chi_p(u)\beta(du)$$

$$= \chi_p(\varphi(s))V_sU_p \; .$$

Similar method shows that (c) implies (a)

Q.E.D.

For any measure μ on $\mathfrak{A}$ we write μ_t for the measure $\mu_t(A) = \mu(A+\varphi(T))$, $A \in \mathfrak{A}$.

Definition 2.1. We say that a non-negative measure μ on $\mathfrak{A}$ is quasi-invariant under φ or $\varphi(R)$-quasi-invariant if $\mu(A) > 0$ implies $\mu(A+\varphi(t)) > 0$ for all $t \in R$. It is called invariant under φ or $\varphi(R)$-invariant if $\mu(A) = \mu(A+\varphi(t))$ for all $t \in R$, $A \in \mathfrak{A}$.

Definition 2.2. A finite measure μ is said to translate continuously in the direction of φ provided $\|\mu_t - \mu\| \to 0$ as $t \to 0$, where $\| \ \|$ denotes the total variation norm.

These definitions are taken from K. deLeeuw and I. Glicksberg [1]. They have shown that if μ is quasi-invariant under φ then μ translates continuously in the direction of φ (see p. 184 of [1]).

Let μ be a finite regular measure on T which is quasi-invariant under φ. It is easy to check that the function $g(.,.)$ defined by $g(t,x) = \frac{d\mu_t}{d\mu}(x)$ satisfies the functional equation

$$g(t+s,x) = g(t,s)g(s,x+\varphi(t)) \quad \text{a.e. } [\mu]. \tag{2.3}$$

The functions satisfying (2.3) are important enough to deserve a special name. Let μ be a fixed positive measure on $\mathfrak{A}$ quasi-invariant under φ.

Definition 2.3. A jointly measurable function $A(.,.)$ on $R \times T$ is called a cocycle if it satisfies the functional equation

$$A(t+s,x) = A(t,x)A(s,x+\varphi(s)) \quad \text{a.e. } [\mu]. \tag{2.4}$$

It is called unitary cocycle if $|A(t,s)| = 1$ a.e. $[\mu]$. The set of μ measure zero where (2.4) does not hold may depend on t and s.

Definition 2.4. A cocycle $A(.,.)$ is called a coboundary if there exists a measurable function $B(.)$ on T such that

$$(2.5) \qquad A(t,x) = \frac{B(x+\varphi(t))}{B(x)} \qquad \text{a.e. } [\mu] .$$

Remark It is easy to see that for any non-vanishing measurable function $B(.)$ on T the function $A(t,x) = \frac{B(x+\varphi(t))}{B(x)}$ is a cocycle. We conclude this section by giving a solution to the equation of theorem 2.1.

Theorem 2.2. A group of unitary operators V_t, $t\in R$, on $L_2(T,\mu)$ satisfies

$$(2.6) \qquad V_s U_p = \chi_p(\varphi(s)) U_p V_s$$

if and only if

(a) μ is a quasi-invariant under φ

(b) there exists a unitary cocycle $A(.,.)$ such that

$$(V_t f)(.) = A(t,.)\sqrt{\frac{d\mu_t}{d\mu}}(.)f(.+\varphi(t)) .$$

Proof Since V_t, $t\in R$, satisfies (2.6) by theorem 2.1 (b) $V_t\beta(A)f = \beta(A+\varphi(t))V_t f$. Take $f = 1$, then $V_t I_A = I_{A+\varphi(t)} V_t 1 = B(t,.)$. Now V_t is a unitary operator so

$$\int_T |I_A(x)|^2\mu(dx) = \mu(A) = \int_{A+\varphi(t)} |B(t,x)|^2\mu(dx) = \int_{A+\varphi(t)} \frac{d\mu_t}{d\mu}(x)\mu(dx).$$

It follows from this that $\mu(A) > 0$ implies $\mu(A+\varphi(t)) > 0$ for all $t\in R$, i.e., μ is quasi-invariant under φ. Further the relation holds for all $A\in\mathfrak{A}$ so we must have $|B(t,x)|^2 = \frac{d\mu_t}{d\mu}(x)$ a.e. $[\mu]$. Hence $B(t,x) = A(t,x)\sqrt{\frac{d\mu_t}{d\mu}}(x)$, where $A(t,x)$ is a $R\times T$ measurable function of absolute value 1. (The joint measurability of $A(.,.)$ is a consequence of the strong continuity of V_t, $t\in R$.) Thus we have $V_t I_A = I_{A+\varphi(t)}A(t,.)\sqrt{\frac{d\mu_t}{d\mu}}(.)$. So for any $f\in L_2(T,\mu)$ we have

$$(V_t f)(.) = A(t,.)\sqrt{\frac{d\mu_t}{d\mu}}\, f(.+\varphi(t)).$$

Now

$$V_{t+s}1 = V_t V_s 1 = V_t A(s,.)\sqrt{\frac{d\mu_t}{d\mu}}$$

$$= A(t,.)A(s,.+\varphi(t))\sqrt{\frac{d\mu_s}{d\mu}}\,(.+\varphi(t))\sqrt{\frac{d\mu_t}{d\mu}}\,(.)$$

$$= A(t,.)A(s,.+\varphi(t))\,\frac{d\mu_{s+t}}{d\mu}\,(.)$$

$$= A(t+s,.)\,\frac{d\mu_{t+s}}{d\mu}\,(.), \qquad \text{a.e. } [\mu].$$

The third equality above is due to relation (2.3). It follows that $A(t+s,.) = A(t,.)A(s,.+\varphi(t))$ a.e. $[\mu]$ and thus $A(.,.)$ is a cocycle which is obviously unitary. Conversely suppose that μ is quasi-invariant under φ and $A(.,.)$ is a unitary cocycle. Let

$$(V_t f)(.) = A(t,.)\sqrt{\frac{d\mu_t}{d\mu}}\, f(.+\varphi(t)),$$

then

$$\int_T |V_t f|^2(x)\mu(dx) = \int_T |f(x+\varphi(t))|^2\,\frac{d\mu_t}{d\mu}\,(x)\mu(dx) = \int_T |f(x)|^2\mu(dx)\ .$$

So V_t is a unitary operator. Also,

$$(V_{t+s}f)(.) = A(t+s,.)\sqrt{\frac{d\mu_{t+s}}{d\mu}(.)}f(.+\varphi(s+t))$$

$$= A(t,.)\sqrt{\frac{d\mu_{t+s}}{d\mu}}\,(.)f(.+\varphi(t+s)$$

$$= A(t,.)\sqrt{\frac{d\mu_t}{d\mu}}\,A(s,.+\varphi(t))\sqrt{\frac{d\mu_s}{d\mu}}\,(.+\varphi(t))f(.+\varphi(s)+\varphi(t))$$

$$= (V_t V_s f)(.).$$

So that V_t, $t\in R$, is a group of unitary operators. It is strongly continuous because $A(.,.)$ is jointly measurable and μ translates continuously in the direction of φ. q.e.d.

. . .

<u>Remark</u> The idea of the above proof is similar to theorem 3.1 of F. Forelli [3]. The above theorem remains true even when μ is infinite but σ-finite and regular measure on T.

Section 3. In this section we give different types of measures on T quasi-invariant under φ. We shall also show that each of these measures has an equivalent σ-finite measure which is invariant under φ, where equivalence of measure is defined as to be the mutual absolute continuity.

Type I. Haar measure on T is invariant under translation by any member of T, hence in particular it is invariant under translation by members of $\varphi(R)$. Any measure μ on $\mathfrak{B}$ equivalent to the Haar measure on T is quasi-invariant under φ.

Type 2. Let $x+\varphi(R)$ be a coset of $\varphi(R)$ in T. By linear measure ℓ_x on $x+\varphi(R)$ we mean the measure defined by $\ell_x(x+\varphi\ (E)) = \ell(E)$, where ℓ is the Lebesgue measure on the real line and E is a Borel subset of R. We shall let ℓ_x stand also for the measure on T defined by $\ell_x(B) = \ell_x(B \cap \{x+\varphi(R)\})$, $B \in \mathfrak{B}$. It is obvious that the linear measure ℓ_x on $x+\varphi(R)$ is invariant under φ and σ-finite. Let μ be any measure on T equivalent to ℓ_x for some x. Then μ is quasi-invariant under φ.

Type 3 (a). Let $x_1, x_2, \ldots$ be a countable number of distinct points of T such that $x_i+\varphi(R)$, $x_j+\varphi(R)$ are distinct cosets if $i \neq j$. For each i, let μ_i be a finite measure on T equivalent to the linear measure on $x_i+\varphi(R)$. Let $\sum_{i=1}^{\infty} \mu_i(x_i+\varphi(R)) < \infty$. Then the measure $\sum_{i=1}^{\infty} \mu_i$ is $\varphi(R)$-quasi-invariant measure on T. The measure $\sum_{i=1}^{\infty} \ell_{x_i}$ is a σ-finite measure invariant under φ and equivalent to $\sum_{i=1}^{\infty} \mu_i$.

The idea of the following construction was suggested to us by Professor R. V. Chacon.

Type 3 (b). Let C denote the circle group. Let S be a perfect set in $C \times \{0\}$ of measure zero (with respect to the Haar measure on

$C \times \{0\}$). It is possible to choose S in such a way that for distinct points $x, y \in S$, the cosets $x+\varphi(R)$, $y+\varphi(R)$ are distinct (see [2]). Let ν be a non-atomic finite measure supported on S. For each $x \in S$, let μ_x be a measure on $x+\varphi(R)$ such that

(a) μ_x is equivalent to the linear measure on $x+\varphi(R)$

(b) $\mu_x(x+E) = \mu_y(y+E)$, $E \subset \varphi(R)$.

It can be shown that under conditions (a) and (b), $\mu_x(A)$ is a measurable function in x for each fixed $A \in \mathfrak{A}$. The essential steps for proving this are as follows: let $a < x < b$, $c < y < d$ be two open intervals in C. Let $A = \{(x,y);\ a < x < b,\ c < y < d\}$ be an open rectangle in T. It can be easily checked that $\mu_x(A)$ is continuous in x for A of the above type so that it is also measurable in x. Next the collection of sets $A \in \mathfrak{A}$ for which $\mu_x(A)$ is measurable in x form a σ-algebra. Hence $\mu_x(A)$ is measurable in x for every $A \in \mathfrak{A}$. Now write $\mu(A) = \int_S \mu_x(A)\nu(dx)$. Clearly μ is a regular finite measure on T. Further it is quasi-invariant under φ for the following reason. Suppose $\mu(A) > 0$. Then $\nu(x\colon \mu_x(A) > 0) > 0$. But each μ_x is quasi-invariant under φ. So $\nu(\{x\colon \mu_x(A+\varphi(t)) > 0\}) > 0$. Consequently $\mu(A+\varphi(t) = \int_S \mu_x(A+\varphi(t))\nu(dx) > 0$, proving the $\varphi(R)$-quasi-invariance of μ. The measure μ constructed above is singular with respect to the Haar measure on T and $\mu(x+\varphi(R)) = 0$ for each $x \in T$. The following general case includes by type 3 (a) and (b) as special cases.

<u>Type 3 (c)</u>. Let S be a measurable set in $C \times \{0\}$ such that distinct points of S belong to distinct cosets of $\varphi(R)$. Let $S = \bigcup_{k=1}^{\infty} S_k$ where each S_k is compact. For each $x \in S$, let μ_x be a finite measure on T such that

(a) μ_x is equivalent to the linear measure ℓ_x

(b) $\mu_x(A)$ is measurable in x for each fixed $A \in \mathfrak{M}$. Let ν be a finite regular measure supported on S. Then the measure μ defined by

$$(3.1) \qquad \mu(A) = \int_T \mu_x(A)\nu(dx), \qquad A \in \mathfrak{M}$$

is quasi-invariant under φ.

We have shown already that measures of type 1, 2 and 3 (a) have equivalent σ-finite measure invariant under φ. It is less obvious that measure of type 3 (c) has an equivalent σ-finite measure invariant under φ. We shall show this presently but first we obtain some preliminary result which shall be used in sequel.

Let S be a subset of $C \times \{0\}$ such that

(a) $S = \bigcup_k S_k$, each S_k compact

(b) if x,y are distinct points of S then $x+\varphi(R)$ $y+\varphi(R)$ are distinct cosets of $\varphi(R)$.

Let J be the map of $S \times R$ onto $S+\varphi(R)$ defined by $J(x,r) = x+\varphi(r)$.

Lemma 3.1. J is one-one and onto with range $S+\varphi(R)$. J and J^{-1} are both measurable. (Here measurable subsets of $S+\varphi(R)$ are induced by $\mathfrak{M}$.)

Proof J is one-one and onto with range $S+\varphi(R)$ because distinct points of S belong to distinct cosets of $\varphi(R)$. Further J is continuous map from $S \times R$ onto $S+\varphi(R)$, hence J is measurable. Now consider J^{-1}: $S+\varphi(R) \to S \times R$. Let $A \subset S$, $B \subset R$ be compact sets. Since φ is continuous $\varphi(B)$ is compact. Now $(J^{-1})^{-1}(A \times B) = J(A \times B) = A+\varphi(B)$ is compact, hence a measurable subset of $S+\varphi(R)$. Now $\mathfrak{F} = \{B: J(B)$ is measurable in $S+\varphi(R)\}$ is a σ-algebra which contains sets of the type $A \times B$, $A \subset S$, $B \subset R$ being compact sets. Hence J^{-1} is also measurable. q.e.d.

Let μ be a measure of type 3 (c). Let $A(.,.)$ be a finite valued function on $R \times T$ which is a jointly measurable cocycle, i.e., it satisfies the equation $A(t+s,y) = A(t,y)A(s,y+\varphi(t))$ a.e. $[\mu]$. It is easy to see from this that for a.e. $x \in S$ with respect to ν measure $A(t+s,x+\varphi(u)) = A(t,x+\varphi(u))A(s,x+\varphi(u)+\varphi(s))$ a.e. u $[\mu_x]$.

Lemma 3.2. The cocycle $A(.,.)$ is a coboundary.

Proof We must show that there exists a measurable function $B(.)$ on T such that $A(t,y) = B(y+\varphi(t))B^{-1}(y)$. $A(.,.)$ is measurable on $R \times T$. A point $y \in S+\varphi(R)$ has unique representation $y = x+\varphi(r) = J(x,r)$ $x \in S$, $r \in R$. Consider

$$(3.2) \qquad C(t,x,r) = A(t,J(x,r)) = A(t,x+\varphi(r)).$$

Since J is measurable map from $S \times R \to T$, $C(t,x,r)$ is measurable in (t,x,r). Further

$$(3.3) \qquad \begin{aligned} C(t+s,x,r) &= A(t+s,x+\varphi(r)) \\ &= A(t,x+\varphi(r))A(s,x+\varphi(r+t)) \\ &= C(t,x,r)C(s,x,r+t) \qquad \text{a.e. } (x,r)\ [\nu \times \ell]. \end{aligned}$$

Further both sides are jointly measurable in all four variables t.s.x. and r. Hence by Fubini theorem there exists r_0 such that $A(t+s,x+\varphi(r_0)) = A(t,x+\varphi(r_0))A(s,x+\varphi(r_0)+\varphi(t))$ a.e. t,s,x. Now the map J^{-1} is measurable, hence $B(\varphi(t)+x+\varphi(r_0)) = A(t,x+\varphi(r_o)) = A(J^{-1}(x+t+\varphi(r_0))$ is a measurable function on $S+\varphi(R)$. Further by (3.3) $A(s,x+\varphi(r_0)+\varphi(t)) = \dfrac{B(x+\varphi(r_0)+\varphi(x))}{B(x+\varphi(r_0))}$. Thus A is a coboundary. q.e.d.

Let μ be the measure defined by 3.1. Then μ is quasi-invariant under φ. By (2.3) $g(t,s) = \dfrac{d\mu_t}{d\mu}(x)$ is a cocycle. By lemma 3.2

it is a coboundary. Let $g(t,x) = \frac{f(y+\varphi(t))}{f(y)}$ a.e. $[\mu]$ for some measurable function f on T. Since f is finite and non-zero almost everywhere $[\mu]$; f is finite and non-zero a.e. $[\mu]$. The following theorem is taken from [8] (Theorem 4.1).

Theorem 3.1. $m(A) = \int_A \frac{1}{f(y)} \mu(dy)$ is a σ-finite measure invariant under φ and equivalent to μ.

Proof Since f is finite and non-zero a.e. $[\mu]$, m is σ-finite and equivalent to μ. Next

$$m(A+\varphi(t)) = \int_{A+\varphi(t)} \frac{1}{f(y)} \mu(dy) = \int_A \frac{1}{f(y+\varphi(t))} \mu_t(dy)$$

$$= \int_A \frac{1}{f(y+\varphi(t))} \frac{d\mu_t}{d\mu} (y)\mu(dy) = \int_A \frac{1}{f(y+\varphi(t))} \frac{f(y+\varphi(t))}{f(y)} \mu(dy)$$

$$= \int_A \frac{1}{f(y)} \mu(dy) = m(A) .$$

Hence m is invariant under φ. q.e.d.

It can be shown that m is equivalent to the measure m' given by $m'(A) = \int_T \ell_x(A)\nu(dx)$.

Type 4. Let μ_1 be a measure of type 1, μ_2 of type 3 (c). Then $\mu_3 = \mu_1+\mu_2$ is again $\varphi(R)$ quasi-invariant measure. Also μ_3 has an equivalent σ-finite measure invariant under φ since μ_1 and μ_2 have.

Definition 3.2. A $\varphi(R)$-quasi-invariant measure is called ergodic if $\mu(E\Delta(E+\varphi(t))) = 0$ for all t implies $\mu(E) = 0$ or $\mu(T-E) = 0$; E being in $\mathfrak{A}$.

Measures of types 1 and 2 are ergodic. The following three problems which are connected with each other, are, as far as we know, unresolved.

(1) Does there exist ergodic measure quasi-invariant under φ which is not of type 1 or 2?

(2) Does there exist a $\varphi(R)$ quasi-invariant regular measure which does not belong to any of the above types?

(3) Does there exist a measure quasi-invariant under φ for which there is no equivalent σ-finite measure invariant under φ?

It is possible that generalizations of results in ([10]) might give solutions to these problems.

Section 4. Quasi-invariant and ergodic measures were introduced in sections 2 and 3 with regard to the subgroup $\varphi(R)$. We need similar definitions for measures on the real line with regard to the subgroup L.

Definition 4.1. A measure μ on $\mathcal{B}$ is called L-quasi-invariant if $\mu(A) > 0$ implies $\mu(A+p) > 0$ for all $p \in L$ and for all Borel subsets $A \in \mathfrak{A}$.

Definition 4.2. An L-quasi-invariant measure μ on $\mathfrak{A}$ is called ergodic if $\mu(A \Delta A+p) = 0$ for all $p \in L$ implies that $\mu(A) = 0$ or $\mu(R-A) = 0$; A being in $\mathfrak{A}$.

The object of this section is to establish a connection between $\varphi(R)$-quasi-invariant measures on T and ergodic L-quasi-invariant measures on R. We shall also show how an unresolved problem about ergodic measures on R is connected to the unresolved problem of evanescent processes. First we state a few elementary results about spectral measures on R. Let E be a spectral measure on $\mathcal{B}$ having as values Hermitian projections in a separable Hilbert space. Let $\varphi_1, \varphi_2, \ldots$ be a complete orthonormal set in H. Let μ_E be the measure on $\mathcal{B}$ defined by $\mu_E(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} (E(A)\varphi_n, \varphi_n)$. It is known and easy to show that $\mu_E(A) \neq 0$ if and only if $E(A) \neq 0$.

Definition 4.3. A spectral measure E on $\mathcal{B}$ is called L-stationary if and only if there exists a group of unitary operator U_p, $p \in L$, such that $U_p E(\sigma) U_{-p} = E(\sigma+p)$ for all $\sigma \in \mathcal{B}$.

Lemma 4.1. E is L-stationary then μ_E is L-quasi-invariant.

Proof Let $\mu_E(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} (E(A)\varphi_i, \varphi_i) > 0$. Then $E(A) \neq 0$. Now U^p is unitary so $E(A+p) = U_p E(A) U_{-p} \neq 0$. Hence

$$\mu_E(A+p) = \sum_{i=1}^{\infty} \frac{1}{2^n} (E(A+p)\varphi_i, \varphi_i) \neq 0. \qquad \text{q.e.d.}$$

. . .

It is obvious that if S is a support of μ_E then $\mu_E(R-S) = 0$ as well as $E(R-S) = 0$.

Definition 4.3. An L-stationary spectral measure E on $\mathcal{B}$ is called ergodic if $E(\sigma) = E(\sigma+p)$ for all $p \in L$ implies $E(\sigma) = 0$ or $E(R-\sigma) = 0$.

Lemma 4.2. An L-stationary spectral measure E is ergodic if and only if μ_E is ergodic.

Proof Let E be ergodic. Let $\sigma \in \mathcal{B}$ such that $\mu_E(\sigma \,\Delta\, \sigma+p) = 0$ for all $p \in L$. Then $E(\sigma \,\Delta\, \sigma+p) = 0$ so that $E(\sigma) = E(\sigma \cap \sigma+p) = E(\sigma+p)$ for $p \in L$. But E is ergodic. Hence $E(\sigma) = 0$ or $E(R-\sigma) = 0$ which is the same as $\mu_E(\sigma) = 0$ or $\mu_E(R-\sigma) = 0$. Hence μ_E is ergodic.

Now suppose μ_E is ergodic and let σ be such that $E(\sigma) = E(\sigma+p)$ for all $p \in L$. Now

$$E(\sigma) = E(\sigma+p) = E(\sigma \vee (\sigma+p \setminus \sigma)) = E(\sigma) + E(\sigma+p \setminus \sigma).$$

So $E(\sigma+p \setminus \sigma) = 0$. Likewise $E(\sigma \setminus \sigma+p) = 0$. Hence $\mu_E(\sigma \,\Delta\, \sigma+p) = 0$ for all $p \in L$. But μ_E is ergodic so $\mu_E(\sigma) = 0$ or $\mu_E(R-\sigma) = 0$. Consequently $E(\sigma) = 0$ or $E(R-\sigma) = 0$, i.e., E is ergodic. q.e.d.

. . .

Theorem 4.1. Let μ be a $\varphi(R)$-quasi-invariant ergodic measure on $\mathfrak{A}$. Let U_p, $p \in L$, be the group of unitary operators in $L_2(T,\mu)$ defined by $U_p f = \chi_p f$, $f \in L_2(T,\mu)$. Let E be a spectral measure on $\mathcal{B}$ with Hermition projections in $L_2(T,\mu)$ as values such that for all $p \in L$, $U_p E(\sigma) U_{-p} = E(\sigma+p)$. Then E and μ_E are both ergodic.

Proof In view of lemma 4.2 it is enough to show that E is ergodic. Let σ be a Borel subset such that $E(\sigma) = E(\sigma+p)$ for all $p \in L$. Then $E(\sigma) = U_p E(\sigma) U_{-p}$. Hence $E(\sigma) L_2(T,\mu)$ is invariant under χ_p, $p \in L$. By (B) of section 2, $E(\sigma)H = L_2(T,\mu|_B)$ where B is a Borel set in T. Now the spectral measure $E_\sigma(A) = E(A \cap \sigma)$ is L-stationary for $U_p E_\sigma(A) U_{-p} = U_p E(\sigma \cap A) U_{-p} = E(\sigma \cap A+p) = E((\sigma+p) \cap (A+p)) = E(\sigma \cap A+p)) = E_\sigma(A+p)$. Hence by theorem 2.1 (i) $\mu|_B$ is quasi-invariant. But μ is ergodic. Hence $\mu = \mu|_B$ or $\mu|_B = 0$. It follows that $E(\sigma) = E(R)$. q.e.d.

. . .

We know only two types of measures on R that are L-quasi-invariant and ergodic. These are

(i) Any measure mutually absolutely continuous with respect to the Counting measure on a fixed coset $x+L$ of R.

(ii) Any measure mutually absolutely continuous with respect to the Lebesgue measure on R.

There are many L-quasi-invariant measures on $\mathcal{B}$ that are non-atomic and singular with respect to the Lebesgue measure on R. A way of constructing such measures is as follows: let μ be a non-atomic measure on R singular with respect to the Lebesgue measure on R. Let $p_0, p_1, p_2, \ldots$ be a denumeration of L. Let

$\nu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(B+p_n)$. Then it is easy to check that μ is L-quasi-invariant and obviously singular with respect to the Lebesgue measure on R. But we do not know if such a measure can be chosen to be ergodic. We establish now a connection of this problem with the problem of evanescent processes. Let μ be a finite regular measure on T absolutely continuous with respect to the Lebesgue measure on T. Let $\mathfrak{M}_t$ = subspace of $L_2(T,\mu)$ spanned by $\{\chi_p : p \le t, p \in L\}$. Helson and Lowdenslager [6] have shown that either $\mathfrak{M}_t = \mathfrak{M}_{t'}$ for all t,t' or else $\mathfrak{M}_t \subsetneqq \mathfrak{M}_{t'}$ $(t < t')$ and $\bigcap_t \mathfrak{M}_t = \{0\}$.

Suppose this second condition is given to hold. Write

$\mathcal{R}_0 = \bigcap_{t>0} \mathfrak{M}_t \ominus \mathfrak{M}_0$. The problem of evanescent processes ([6] p. 183) is to decide whether $\mathcal{R}_0$ is sometimes or always non-trivial and what properties of μ decide between the two cases if they can both occur. Helson and Lowdenslager have shown that if $\log \frac{d\mu}{dm} \in L_1(T,m)$, m being the Haar measure on T, then $\mathcal{R}_0 \neq \{0\}$. ([6], theorem 5). Now we know that μ, being equivalent to m, is ergodic. Hence the spectral measure E defined by $E(a,b]$ = orthogonal projection on $\mathcal{H}_b \ominus \mathcal{H}_a$ is ergodic with respect to the subgroup L. A positive solution to the preceding question concerning L-ergodic measures on R will show that μ_E is either equivalent to the counting measure on a fixed coset of L or the Lebesgue measure on R.

Remarks Ergodicity in the context of "analytic measures" were first considered by F. Forelli [3]. Our theorem 4.1 is similar to a theorem due to Xio Dua-Xing [11].

Section 5. It is well known that the prediction theory of discrete parameter univariate weakly stationary stochastic process is intimately connected with the theory of H_2 and H_1 functions on the circle. The analogue of such functions for our context are special kind of signed measures called "analytic measures".

Definition 5.1. A finite signed measure μ on T is called analytic if $\int \chi_p(x)\mu(dx) = 0$ for $p < 0$, $p \in L$.

For any signed measure μ we shall write $|\mu|$ for the total variation measure of μ. Obviously $\mu(dx) = e(x)|\mu|(dx)$ where $e(.)$ is a measurable function of absolute value 1. We consider the subspaces $\mathcal{H}_s$ of $L_2(T,|\mu|)$ generated by $\{\chi_p(.)e(.): p \le s\}$ for each $s \in R$. They have the following properties

(5.1) (a) $\mathcal{H}_s \subset \mathcal{H}_u$ if $s < u$

(b) $\bigvee_{-\infty<s<\infty} \mathcal{H}_s = L_2(T,|\mu|)$

(c) $\bigcap_s \mathcal{H}_s = \{0\}$.

(a) and (b) are obvious; only property (c) needs a proof. Consider $\int_T \chi_p(x)\chi_q(x)|\mu|(dx)$. If $q \le -t$ then for $p < t$ we have $\int_T \chi_p(x)\chi_q(x)e(x)|\mu|(dx) = 0$ by analylicity of μ. Since $\mathcal{H}_{-t}$ is spanned by $\{\chi_p(.)e(.): p \le -t\}$, we have for every $f \in \mathcal{H}_{-t}$, $\int_T \chi_p(x)f(x)e(x)|\mu|(dx) = 0$ for $p \le t$. Let $f \in \bigcap_{-\infty<t<\infty} \mathcal{H}_t$. Then $f \in \mathcal{H}_{-t}$ for every t. Hence we get $\int_T \chi_p(x)f(x)e(x)|\mu|(dx) = 0$ for all p. This implies $f = 0$ a.e. $[|\mu|]$ proving (c).

Let $E(a,b]$ = orthogonal projection on $\mathcal{H}_b \ominus \mathcal{H}_a$ and let U_p, $p \in L$, be the unitary operator on $L_2(T,\mu)$ defined by $U_p f = \chi_p f$. E is an L-stationary spectral measure in $L_2(T,\mu)$. By theorem 2.2,

μ is a quasi-invariant under φ. This proves for the torus the main theorem of de Leeuw and Glicksberg [1].

Theorem 5.1. If μ is analytic then $|\mu|$ is quasi-invariant under φ.

Let ν be a probability measure on T then $X_p = \chi_p$ is a stationary stochastic process defined on $(T, \mathfrak{A}, \nu)$. Suppose ν is the total variation of an analytic measure μ so that $\mu(dx) = e(x)\nu(dx)$ for some function of absolute value 1 and $\int \chi_p(x)e(x)\nu(dx) = 0$ for $p \le 0$. By making use of the mapping $\chi_p \to e\chi_p$ it can be shown that X_p is purely non-deterministic. This gives

Theorem 5.2. If ν is total variation of analytic measure then X_p, $p \in L$, is purely non-deterministic.

In analogy with the classical case of prediction problem one can ask if the following converse of the above theorem is true.

"If X_p, $p \in L$, is purely non-deterministic, then ν is the total variation of an analytic measure."

An affirmative answer to the problem of evanescent processes will solve this question for the case when ν is absolutely continuous with respect to the Haar measure. In sections 6 and 7, we shall settle this question for the case when ν is of type 3 (c). An example of analytic measure whose total variation is of type 3 (c) is the following.

Let S be a Borel measurable subset of $C \times \{0\}$ such that distinct points of S belong to distinct cosets of $\varphi(R)$. Let ν be a finite positive regular measure on S. Let μ_x be the measure on $x+\varphi(R)$ defined by $\mu_x(x+\varphi(B)) = \lambda(B)$, where λ is an analytic measure on the real line. Then $\mu(A) = \int \mu_x(A)\nu(dx)$ is an analytic measure on T with $|\mu|(A) = \int |\mu_x|(A)\nu(dx)$.

Section 6. In this section we give some auxillary results which shall be used in the next section. Let ℓ denote the Lebesgue measure on the real line. Let $L_2(R)$ denote the set of all Borel measurable functions square integrable with respect to ℓ. Let N be the cardinal number $\leq \aleph_0$. We shall write $L_2^N(R) = \{(f_1,f_2,f_3,\ldots) : \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |f_i(x)|^2 dx < \infty\}$. Consider the strongly continuous group of unitary operators Q_t, $t\in R$, on $L_2^N(R)$ defined by the translations $Q_t(f_1,\ldots,f_n,\ldots) = f_1(.+t),f_2(.+t),\ldots)$. By Stones theorem Q_t has a representation

$$Q_t = \int_{-\infty}^{\infty} e^{it\lambda} E(d\lambda)$$

with spectral measure E on $\mathcal{B}$. Let $H_2 = \{f: f\in L_2(R), \hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx = 0$ for a.e. $t \leq 0\}$. Let $H_2^N = \{(f_1,f_2,\ldots)$: each $f_i \in H_2$, $\int_{-\infty}^{\infty} \sum_{i=1}^{\infty} |f_i|^2 < \infty\}$. The following result is known.

Theorem 6.1. $E(-\infty,0]L_2^N(R) = H_2^N$.

Now let S be a Borel subset of R and ν a finite regular measure on S. Let denote the product σ-algebra in $S \times R$. We shall write $L_2(S \times R)$ for $L_2(S \times R, \ ,\nu \times \ell)$.

Definition 6.1. $f\in L_2(S \times R)$ is called analytic if for a.e. x with respect to ν measure $f(x,.) \in H_2$. $\mathcal{H}_2$ shall denote the class of functions in $L_2(S \times R)$ which are analytic.

$\mathcal{H}_2$ is a closed subspace of $L_2(S \times R)$. Let $\overline{V}_t$ be defined by $(\overline{V}_t f)(.,.) = f(.,.+t)$, $f\in L_2(S \times R)$. Let $\varphi_1,\varphi_2,\ldots$ be a complete orthonormal in $L_2(S,\nu)$. Write for $f\in L_2(S \times R)$ $Wf = (f_1,f_2,\ldots)$, where $f_i \in L_2(R)$ defined by $f_i(.) = \int_S f(x,.)\varphi_i(x)\nu(dx)$. Then W is an invertible isometry from $L_2(S \times R)$ onto $L_2^N(R)$ such that

(a) $W^{-1}Q_tW = \overline{V}_t$

(b) $W\mathcal{H}_2 = H_2^N$

(c) $W^{-1}E(B)W = F(B)$, where F is the spectral resolution

corresponding to $\overline{V}_t$; $\overline{V}_t = \int_{-\infty}^{\infty} e^{it\lambda}F(d\lambda)$.

<u>Theorem 6.2</u>. $F(-\infty,0]L_2(S \times R) = \mathcal{H}_2$.

<u>Proof</u> $W^{-1}E(-\infty,0]WL_2(S \times R) = W^{-1}E(-\infty,0]L_2^N(R) = W^{-1}H_2^N = \mathcal{H}_2$. q.e.d.

Section 7. In this section we shall describe the family of subspaces $\mathcal{M}_p$, $p \in L$, satisfying the conditions (i) (ii) and (iii) of section 2 for measures of type 3 (c). Let S be a Borel subset of $C \times \{0\}$ such that distinct points of S belong to distinct cosets of $\varphi(R)$. Let ν be a finite regular measure on S. For each $x \in S$ let ℓ_x denote the linear measure on $x+\varphi(R)$. Consider $\mu(A) = \int \ell_x(A)\nu(dx)$. μ defines a regular σ-finite measure on T invariant under φ. Consider $L_2(T,\mu)$. Let V_t, $t \in R$, denote the strongly continuous group of unitary operators on $L_2(T,\mu)$ defined by $(V_t f)(.) = f(.+\varphi(t))$. Let its spectral resolution be

$$V_t = \int_{-\infty}^{\infty} e^{itu} G(du) .$$

We know that $S+\varphi(R)$ is a support of μ and every point $y \in S+\varphi(R)$ has a unique representation $y = x+\varphi(r)$, $x \in S$, $r \in R$. (This is because distinct points of S belong to distinct cosets of $\varphi(R)$.) It can be shown that $f \in L_2(T,\mu)$, then $f \in L_2(T,\ell_x)$ for almost every $x \in S$ with respect to ν measure and $\int_T |f(y)|^2 \mu(dy) = \int_S (\int_{-\infty}^{\infty} |f(x+\varphi(t))|^2 dt)\nu(dx)$ which we can also write as $\int_S (\int_T |f(z)|^2 \ell_x(dz))\nu(dx)$.

Definition 7.1. A function $f \in L_2(T,\mu)$ is called analytic if and only if for almost every $x \in S$ with respect to ν measure $f(x+\varphi(.)) \in H_2$, i.e., if for almost every $x \in S$ $\int_{-\infty}^{\infty} e^{itr} f(x+\varphi(r))dr = 0$ for a.e. $t < 0$. $H_2(T,\mu)$ denotes the class of analytic functions in $L_2(T,\mu)$.

$H_2(T,\mu)$ is a closed subspace of $L_2(T,\mu)$.

Theorem 7.1. $G(-\infty,0]L_2(T,\mu) = H_2(T,\mu)$.

Proof Consider the map J defined in section 3. $J(x,r) = x+\varphi(r)$. The map J defines an invertible isometry J_1 from $L_2(S \times R, \nu \times L)$ onto $L_2(\mu)$

$$J_1 f(x,r) = f(x+\varphi(r)).$$

J_1 has the following properties

(a) $J_1 \overline{V}_t J_1^{-1} = V_t$, where $\overline{V}_t$ is as in section 6.

(b) $J_1 \mathcal{H}_2 = H_2(t,\mu)$

(c) $J_1 F(B) J_1^{-1} = G(B)$, $B \in \mathcal{B}$, where F is the spectral resolution of $\overline{V}_t$, $t \in R$.

Now by theorem 6.2 $F(-\infty,0]L_2(S \times R, \nu \times L) = \mathcal{H}_2$. Hence

$$\begin{aligned} G(-\infty,0]L_2(T,\mu) &= J_1 F(-\infty,0] J_1^{-1} L_2(T,\mu) \\ &= J_1 F(-\infty,0] L_2(S \times R, \nu \times L) \\ &= J\mathcal{H}_2 = H_2(T,\mu). \end{aligned}$$

q.e.d.

Let us write $\mathcal{K}_p = \chi_p H_2(T,\mu)$. If U_p be multiplication by χ_p, then $U_p V_s = e^{its} V_s U_p$. Hence by theorem 2.1 (c) $U_p G(B) U_{-p} = G(B+p)$ and $\mathcal{K}_p = G(-\infty,p]L_2(T,\mu)$. So $\bigcap_p \mathcal{K}_p = \{0\}$. Now let a be a measurable function on T of absolute value 1 a.e. $[\mu]$. Consider $\mathcal{H}_0^* = aH_2(T,\mu)$, $\mathcal{H}_p^* = \chi_p \mathcal{H}_0^* = a\mathcal{K}_p$. Let $\mathcal{H}_t^*$ = space spanned by $\bigcup_{p \le t} \mathcal{H}_p$. Then it is easy to check that $\bigcap_{t=-\infty}^{\infty} \mathcal{H}_t^* = \{0\}$ and $\bigcup_{t=-\infty}^{\infty} \mathcal{H}_t^*$ spans $L_2(T,\mu)$. Let $E^*(a,b]$ denote the orthogonal projection on $\mathcal{H}_b^* \ominus \mathcal{H}_a^*$.

Then $E^*(a,b] = aG(a,b]a^{-1}$.

<u>Theorem 7.2</u>. Let $V_t^* = \int_{-\infty}^{\infty} e^{itu} E^*(du)$. Then $(V_t^* f)(.) = a(.+\varphi(t))a^{-1}(.)f(.+\varphi(t))$.

Proof

$$V_t^* f = \int_{-\infty}^{\infty} e^{itu} E^*(du) f = \int_{-\infty}^{\infty} e^{itu} aG(du)a^{-1} f$$

$$= a\int_{-\infty}^{\infty} e^{itu} G(du)a^{-1} f = aV_t a^{-1} f.$$

Hence $(V_t^* f)(.) = a^{-1}(.+\varphi(t)a(.)f(.+\varphi(t)).$ q.e.d.

Following is the main theorem of this section.

Theorem 7.3. Let $\mathcal{H}_t$, $t\in R$, be a family of subspaces in $L_2(T,\mu)$ satisfying the condition (i) (ii) (iii) of section 2. Then $\mathcal{H}_0 = aH_2(T,\mu)$ for some measurable function a of absolute value 1.

Proof Let $E(a,b]$ = orthogonal projection on $\mathcal{H}_b \ominus \mathcal{H}_a$. Write $V_t = \int_{-\infty}^{\infty} e^{itu} E(du)$. By theorem 2.2 there exists a unitary cocycle $A(.,.)$ such that $(V_t f)(.) = A(t,.)f(.+\varphi(t))$.

By lemma 3.2, $A(t,x)$ is a coboundary which we take to be of the form $a(x+\varphi(t)a^{-1}(x)$. Consider the family of subspaces $\mathcal{H}_t^* = \chi_t aH_2(T,\mu)$. They give rise to the spectral measure E^*, $E^*(a,b)$ = orthogonal projection on $\mathcal{H}_b^* \ominus \mathcal{H}_a^*$. Let $V_t^* = \int_{-\infty}^{\infty} e^{itu} E^*(du)$. Theorem 7.2 shows that $V_t^* f = V_t f$ for $f\in L_2(T,\mu)$. Hence $E^* = E$. Therefore $aH_2(T,\mu) = \mathcal{H}_0^* = E^*(-\infty,0]L_2(T,\mu) = E(-\infty,0]L_2(T,\mu) = \mathcal{H}_0$. q.e.d.

Let μ_1 be a measure equivalent to μ. Let $\rho(.) = \frac{d\mu_1}{d\mu}$. Let $\mathcal{H}_t$ be a family of subspaces in $L_2(T,\mu_1)$ satisfying (i) (ii) and (iii) of section 2. Then $\mathcal{H}_t\sqrt{\rho}$ is a family of subspaces in $L_2(T,\mu)$ satisfying the same conditions. Hence by theorem 7.3 $\mathcal{H}_t\sqrt{\rho} = \chi_t aH_2(T,\mu)$ for some function a of absolute value 1. Hence we have

Theorem 7.4. Let $\mathcal{H}_t$, $t \in R$, be a family of subspaces in $L_2(T,\mu_1)$ satisfying the conditions (i) (ii) (iii) of section 2. Then,

$$\mathcal{H}_0 = a \frac{1}{\sqrt{\rho}} H_2(T,\mu).$$

Section 8. Let X_p, $p \in L$, be a weakly stationary stochastic process with $(X_p, X_0) = \int_T \chi_p(u)\mu_1(du)$. Let $\mathcal{M}_t$ be the subspace of $L_2(T,\mu)$ spanned by $\{\chi_p : p \leq t, p \in L\}$.

Theorem 8.1. Suppose μ_1 is type 3 (c). Then $\bigcap_{-\infty<t<\infty} \mathcal{M}_t = \{0\}$ if and only if μ_1 is the total variation of an analytic measure.

Proof We already know from section 5 that if μ_1 is the total variation of an analytic measure then $\bigcap_t \mathcal{M}_t = \{0\}$. Now suppose that $\bigcap_t \mathcal{M}_t = \{0\}$. Then $\mathcal{M}_t$, $t \in R$ satisfy conditions (i) (ii) and (iii) of section 2. Hence by theorem 7.3, $\mathcal{M}_0 = a \frac{1}{\sqrt{\rho}} H_2(T,\mu)$ where μ is an invariant measure equivalent to μ_1 and $\rho = \frac{d\mu_1}{d\mu}$. Hence $a^{-1}\sqrt{\rho}\,\mathcal{M}_0 = H_2(T,\mu)$ so that $a^{-1}\rho \in H_2(T,\mu)$. Consider the measure $\Psi(B) = \int_B a^{-2}(x)\rho(x)\mu(dx)$. Clearly $|\Psi| = \mu_1$. We show that Ψ is analytic

$$\int_T \chi_p(u)\psi(du) \quad = \quad \int_S \Big(\int_{-\infty}^{\infty} \chi_p(x+\varphi(t))a^{-2}(x+\varphi(t))\rho(x+\varphi(t))dt\Big)du.$$

Now $a^{-2}(x+\varphi(.))\rho(x+\varphi(.)) \in H_1$, the usual Hardy class on the line. Hence

$$\int_{-\infty}^{\infty} \chi_p(x+\varphi(t))a^{-2}(x+\varphi(t))\rho(x+\varphi(t))dt = 0 \quad \text{for } p < 0 \text{ so that}$$

$$\int_T \chi_p(u)\psi(du) \quad = \quad 0 \quad \text{for } p < 0. \qquad \text{q.e.d.}$$

Theorem 8.2. Let μ_1 be of type 3 (c). Then X_p, $p \in L$, is purely non-deterministic if an only if $\int_{-\infty}^{\infty} \frac{\log \frac{d\mu_1}{d\mu}(x+\varphi(t))}{1+t^2}\,dt < \infty$ for a.e. x $[\nu]$.

Proof X_p, $p \in L$, is purely non-deterministic if andonly if $\bigcap_{-\infty<t<\infty} \mathcal{M}_t = \{0\}$. Hence by theorem 8.1 $a\sqrt{\rho} \in H_2(T,\mu)$, $|a| = 1$. So $a\sqrt{\rho}(x+\varphi(.)) \in H_2$

for a.e. x [ν]. Hence by Paley-Wiener theorem $\int_{-\infty}^{\infty} \frac{\log\rho(x+\varphi(t))}{1+t^2}\, dt < \infty$

for a.e. x [ν]. Conversely if $\int_{-\infty}^{\infty} \frac{\log\rho(x+\varphi(t))}{1+t^2}\, dt < \infty$ for a.e. x [ν].

Then $\rho(x+\varphi(.)) = a(x+\varphi(.)) \cdot h(x+\varphi(.))$ where $h(x+\varphi(.)) \in H_2$ and a is a function of absolute value 1.

The analytic measure $\Psi(B) = \int_B h(y)\mu(dy)$ has the same total variation as μ_1. Hence by theorem 8.1 X_p, $p \in L$, is purely non-deterministic. q.e.d.

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